Buckling of Orthotropic Sandwich Cylinders Under Axial Compression and Bending

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The buckling analysis reported here is for finite-length sandwich cylinders clamped at both edges and loaded in axial compression, pure bending, or a combination of these loadings. Love's first-approximation shell theory is used, and a Rayleigh-Ritz solution is obtained. For axially compressed sandwich cylinders with clamped boundary conditions, the theory predicts values above published experimental results. For pure bending, predicted results are considerably above the experimental results; however, the experimental data represent a very limited number of specimens. Results for pure bending and combined bending and axial compression are in general agreement with published theoretical results for isotropic thin-walled simply-supported cylinders.

Nomenclature

$\bar{A}_{ij}, \bar{A}_{ij}^k$ = arbitrary dimensionless coefficients of assumed modal functions
A_i = stiffness coefficients (i = 1 to 15)
E, E_s, E_t = Young's modulus, secant modulus, and tangent modulus for an isotropic material
E_x, E_θ = facing elastic moduli in the x and θ directions
$\bar{E}_x, \bar{E}_\theta$ = $E_x/(1 - \nu_{\theta x}\nu_{x\theta}), E_\theta/(1 - \nu_{\theta x}\nu_{x\theta})$
$\bar{e}_{ij}$ = engineering strains
$\tilde{e}_{ij}$ = membrane facing strains
$G_{x\theta}$ = facing shear modulus in x-θ plane
$G_{zx}, G_{\theta z}$ = core shear moduli in the z-x and θ-z planes
h = core half-thickness
K_x, K_θ = shear correction factors ($\approx$ 1 for sandwich with relatively thin facings)
L = shell length; load matrix
$\bar{M}, M'$ = bending moment and bending load factor
$\bar{m}_k$ = number of longitudinal half waves in the assumed modal term
$\bar{N}_i, \bar{N}_{ij}$ = membrane stress resultants
n, n_l = number of circumferential full waves
P, P' = axial load and axial load factor
R = radius of shell middle surface
S = stiffness matrix
t = facing half-thickness
u, v, w = middle-surface displacements in the x, θ, and z directions
V = strain energy
$\bar{V}_1, \bar{V}_2$ = load coefficients
x, θ, z = shell coordinates in the axial, circumferential, and thickness directions
y_i = generalized coordinate
β_x, β_θ = angles of rotation of the normal to the middle surface for the facings in the axial and circumferential directions
ζ = R^{-1}
λ = eigenvalues
ν = plastic value of Poisson's ratio as defined in Eq. (8)
ν_e = elastic value of Poisson's ratio for an isotropic material
$\nu_{x\theta}, \nu_{\theta x}$ = Poisson's ratios defined in Eqs. (7)
σ_{ij} = stress
φ_j = assumed modal functions
Ψ_x, Ψ_θ = angle of rotation of the normal to the shell middle surface for the core in the axial and circumferential directions
ξ_i = physical components of displacement at an arbitrary position

Subscripts
b, c = bending, compression
x, θ, z = coordinates
1, 2 = components of strain energy

Superscripts
c, f = core and facings
i, o = inner and outer facings
u, v, w, x, θ = displacements

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Index categories: Structural Composite Materials (Including Coatings); Structural Design, Optimal; Structural Stability Analysis.

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1. Introduction

BECAUSE of its high structural efficiency, sandwich-type construction with orthotropic composite-material facings is now being considered for many primary structural applications. However, for sandwich-type construction, there are not many analyses available even for the buckling of isotropic shells. Platema[1] provided the most complete survey for this type of construction. The most extensive buckling analysis of axially compressed isotropic sandwich shells is that of Bartelds and Mayers,[2] using a unified theory which covers face-wrinkling (short wavelength) type of buckling as well as general instability. Wang et al.,[3] using a Donnell-type theory and a Galerkin solution, treated the combined loading case for a long cylinder.

For an axially compressed sandwich cylinder with orthotropic facings and core, three basic analyses exist.[4-6] Reference 4 is a large-deflection analysis that has been shown[5] to be physically invalid. Reference 5 is a general instability analysis based on the small-deflection, Donnell-type theory of Stein and Mayers.[7] Reese and Bert[8] investigated this linear analysis[5] using a digital computer and presented a set of approximate design equations. Later Peterson[6] also presented an orthotropic analysis based on the sandwich shell theory of Stein and Mayers.

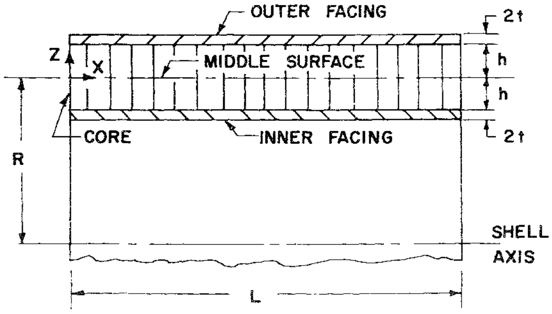

Fig. 1 Shell geometry.

With one exception, Peterson's analysis is identical to the analysis of Ref. 5. The Poisson's ratios for bending and extension are assumed equal in Ref. 5; however, in practice the effect of this assumption is not restrictive.

All other pertinent linear buckling analyses of sandwich cylinders are for special cases: isotropic facings and unidirectional core,[9,10] isotropic facings and orthotropic core,[11,12] and orthotropic facings and rigid core.[13] Most of the previous cylindrical sandwich analyses considered classical simple-support boundary conditions. The rest considered the cylinder to be long, thereby making the boundary conditions unimportant.

The present paper is believed to be the first to consider finite-length cylinders with clamped-support boundary conditions. The loading considered is axial compression, bending, or combined axial compression and bending. The facings and core can be either isotropic or orthotropic.

2. Hypotheses

The analysis is based on the following assumptions:

1) The shell geometry and loading are shown in Figs. 1 and 2.

2) The facings are capable of developing extensional in-surface strain energy and bending strain energy, but not transverse shear strain energy or strain energy from extension in the thickness direction.

3) The facings are linearly elastic of identical construction, and can be orthotropic.

4) The core is capable of developing transverse shear strain energy only, is linearly elastic, and can be orthotropic with respect to its transverse shear properties.

5) The deflections are assumed to be sufficiently small to allow linearization of the strain-displacement relations.

6) The shell thickness is small compared to the shell radius.

7) For the core, lines which are straight and normal to the middle surface before deformation remain straight during deformation but not necessarily normal to the middle surface.

8) For the facings, lines which are straight and normal to the facing-core interface before deformation remain so after deformation. (Reissner's version of Love's first-approximation theory, as given by Kraus,[14] is used in the formulation of the strain equations).

9) The prebuckling stress state is assumed to be a pure membrane one.

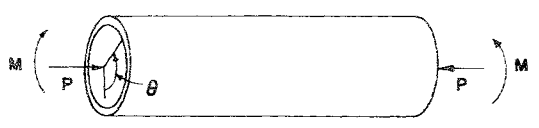

Fig. 2 Configuration and loading.

3. Strain Equations

The strain expressions are

$$e_{xx} = \xi_{x,x} \; ; \; e_{\theta\theta} = (\xi_{\theta,\theta} + \xi_z)/R$$
$$e_{zz} = \xi_{z,z} \; ; \; e_{x\theta} = (\xi_{x,\theta} + R\xi_{\theta,x})/R \tag{1}$$
$$e_{\theta z} = (R\xi_{\theta,z} - \xi_\theta + \xi_{z,\theta})/R \; ; \; e_{zx} = \xi_{z,x} + \xi_{x,z}$$

where the comma denotes partial differentiation. These strain equations can be found by particularizing the Love[15] curvilinear coordinates or from a tensorial approach.[16]

The assumed displacements are as follows: for the core,

$$\xi_x{}^c = u(x, \theta) + z\Psi_x(x, \theta) \; ; \; \xi_\theta{}^c = v(x, \theta) + z\Psi_\theta(x, \theta) \; ; \; \xi_z{}^c = w(x, \theta) \tag{2}$$

for the outer facing,

$$\xi_x{}^0 = u(x, \theta) + h\Psi_x(x, \theta) + (z - h)\beta_x{}^0(x, \theta)$$
$$\xi_\theta{}^0 = v(x, \theta) + h\Psi_\theta(x, \theta) + (z - h)\beta_\theta{}^0 (x, \theta) \; ; \; \xi_z{}^0 = w(x, \theta) \tag{3}$$

for the inner facing,

$$\xi_x{}^i = u(x, \theta) - h\Psi_x(x, \theta) + (z + h)\beta_x{}^i(x, \theta)$$
$$\xi_\theta{}^i = v(x, \theta) - h\Psi_\theta(x, \theta) + (z + h)\beta_\theta{}^i(x, \theta) \; ; \; \xi_z{}^i = w(x, \theta) \tag{4}$$

In addition to the linear strain components that result from substituting Eqs. (2–4) into Eq. (1), the membrane stretching due to displacement in the thickness direction must be taken into account[17]:

$$e_{xx} = (1/2) \; \beta_x{}^2 \; ; \; e_{\theta\theta} = (1/2) \; \beta_\theta{}^2 \; ; \; e_{x\theta} = \beta_x\beta_\theta \tag{5}$$

4. Constitutive Equations

In view of Hypothesis 3, the generalized Hooke's law for the core is as follows:

$$\sigma_{zx}{}^c = e_{zx}{}^c \; G_{zx} \; ; \; \sigma_{\theta z}{}^c = e_{\theta z}{}^c \; G_{\theta z} \tag{6}$$

A generalized state of plane stress is assumed to exist in the facings. The generalized Hooke's law for the facings then becomes

$$\sigma_{xx}{}^0 = \bar{E}_x \; (e_{xx}{}^0 + \nu_{\theta x}e_{\theta\theta}{}^0) \; ; \; \sigma_{\theta\theta}{}^0 = \bar{E}_\theta \; (e_{\theta\theta}{}^0 + \nu_{x\theta} \; e_{xx}{}^0)$$
$$\sigma_{x\theta}{}^0 = G_{x\theta}e_{x\theta}{}^0 \; ; \; \sigma_{xx}{}^i = \bar{E}_x \; (e_{xx}{}^i + \nu_{\theta x}e_{\theta\theta}{}^i) \tag{7}$$
$$\sigma_{\theta\theta}{}^i = \bar{E}_\theta(e_{\theta\theta}{}^i + \nu_{x\theta}e_{xx}{}^i) \; ; \; \sigma_{x\theta}{}^i = G_{x\theta}e_{x\theta}{}^i$$

As a consequence of the symmetry of the stiffness coefficient matrix for an arbitrary elastic material

$$E_x\nu_{\theta x} = E_\theta\nu_{x\theta} \tag{8}$$

5. Energy Formulation

The strain energy for the core is due to transverse shear strain only. Therefore

$$V^c = (1/2) \int_x \int_\theta \int_z (\sigma_{\theta z}{}^c \; e_{\theta z}{}^c + \sigma_{zx}{}^c e_{zx}{}^c)dz \; R \; d\theta \; dx \tag{9}$$

The strain energy for the facings consists of two parts. In addition to the part due to bending $(V_2{}')$, there is a contribution due to the membrane load condition that exists at the onset of buckling (V_1^f). This membrane state of stress is assumed to remain essentially constant during buckling. Then

$$V_1^f = \int_x \int_\theta \int_z [\tilde{\sigma}_{xx}{}^0 \tilde{e}_{xx}{}^0 + \tilde{\sigma}_{x\theta}{}^0 \tilde{e}_{x\theta}{}^0 + \tilde{\sigma}_{xx}{}^i \tilde{e}_{xx}{}^i + \tilde{\sigma}_{x\theta}{}^i \tilde{e}_{x\theta}{}^i] dz\ R\ d\theta\ dx$$

$$V_2^f = (1/2) \int_x \int_\theta \int_z [\ \sigma_{xx}{}^0\ e_{xx}{}^0 + \sigma_{\theta\theta}{}^0\ e_{\theta\theta}{}^0 + \sigma_{x\theta}{}^0 e_{x\theta}{}^0 + \sigma_{xx}{}^i\ e_{xx}{}^i + \sigma_{\theta\theta}{}^i\ e_{\theta\theta}{}^i + \sigma_{x\theta}{}^i\ e_{x\theta}{}^i]\ dz\ R\ d\theta\ dx \quad (10)$$

where $\tilde{\sigma}$ and $\tilde{e}$ are membrane components.

The potential energy due to the external axial compression P and axial bending moment M is equal in magnitude and opposite in sign to the sum of the linear membrane strain terms included in V_1^f. The other linear terms are odd functions of z and the z-integral for the inner facing cancels out the z-integral for the outer facing.[16]

For the external loadings considered, the stress resultants prior to buckling are assumed to be given by the membrane relations[3]

$$\tilde{N}_x = 2\ \tilde{N}_x{}^0 = 2\ \tilde{N}_x{}^i = [\ -(P/2) + M(\cos\theta)/R]/(\pi R)$$
$$\tilde{N}_{x\theta} = 2\ \tilde{N}_{x\theta}{}^0 = 2\ \tilde{N}_{x\theta}{}^i = 0 \quad (11)$$

In order to simultaneously increase the various loads in the combined loading case, it is convenient to assume that P and M can be represented by a common factor λ, such that

$$P = \lambda P' \ ; \ M = \lambda M' \quad (12)$$

If one substitutes the constitutive and strain equations into the combined Eqs. (9) and (10) and integrates the result through the thickness, one obtains the result given in the Appendix, Eq. (A1).

6. Rayleigh-Ritz Method

The Rayleigh-Ritz procedure is started by assuming a series of deflection modal functions of the form

$$y_i = \sum_{j=1}^{\infty} \bar{A}_{ij}\varphi_j(x, \theta) \quad (13)$$

where y_i is the generalized coordinate, $\bar{A}_{ij}$ are the undetermined modal coefficients, and $\varphi_j(x,\theta)$ are the assumed modal functions. If Eqs. (13) are truncated to a given number of terms and then substituted into the total energy expression, Eq. (A1), the resulting equation is second order with respect to the $\bar{A}_{ij}$

$$S(\bar{A}_{ij}\bar{A}_{kl}) - \lambda L(\bar{A}_{ij}\bar{A}_{kl}) = \text{Total Energy} \quad (14)$$

Then the total energy can be minimized with respect to each of the A_{ij}, thus resulting in (ixj) equations which are linear in the modal coefficients $\bar{A}_{ij}$. These (ixj) equations can be written in matrix form as

$$([S] - \lambda[L])\{\bar{A}_{ij}\} = 0 \quad (15)$$

Equation (15) is in standard eigenvalue form. The eigenvalue represents the critical axial load, critical moment or a particular combined state of loading. For the Rayleigh-Ritz method, it is necessary to satisfy exactly the geometric boundary conditions only. However, since it is impossible to utilize an infinite number of terms in the assumed modal functions, the resulting convergence depends on how nearly all of the boundary conditions are satisfied.

7. Boundary Conditions

For the three displacements and two rotations included in this analysis, the subject boundary conditions becomes very complex. For example, a thin-walled cylinder (not a sandwich one) with displacements u, v, w and clamped at both ends has four possible sets of clamped conditions

$$\begin{array}{llll} w = 0 & w,_x = 0 & \sigma_x = 0 & \sigma_{x\theta} = 0 \\ w = 0 & w,_x = 0 & u = 0 & \sigma_{x\theta} = 0 \\ w = 0 & w,_x = 0 & \sigma_x = 0 & v = 0 \\ w = 0 & w,_x = 0 & u = 0 & v = 0 \end{array} \quad (16)$$

A similar set exists for the simple-support case. It is easily seen that the two additional rotations (Ψ_x, Ψ_θ), present in a sandwich shell, greatly amplify the number of possible sets.

For the Rayleigh-Ritz method of solution, the boundary conditions are applied through proper selection of the assumed deflection modal functions. Since these deflection modal functions must normally be limited to a very few terms, the functions should be selected such that only a few terms are needed for convergence to the actual deflected form.

Although modal functions representing any of the various boundary conditions can be applied to Eq. (A1), only one set is considered here: combined bending and axial compression. To the authors' knowledge, the only reported displacement function for the clamped case is the one given by Batdorf et al.[18] for w. A generalized form of this function is used here for w, along with an appropriate set of functions for u, v, Ψ_x, and Ψ_θ

$$w = \sum_k [\bar{A}_{4k}{}^w \cos(n-1)\theta + \bar{A}_{5k}{}^w \cos n\theta + \bar{A}_{6k}{}^w \cos(n+1)\theta] \{\cos(m_k\pi x/L) - \cos[(m_k + 2)\pi x/L]\}$$

$$u = \sum_k [\bar{A}_{4k}{}^u \cos(n-1)\theta + \bar{A}_{5k}{}^u \cos n\theta + \bar{A}_{6k}{}^u \cos(n+1)\theta] \cos(m_k\pi x/L)$$

$$v = \sum_k [\bar{A}_{1k}{}^v \sin(n-1)\theta + \bar{A}_{2k}{}^v \sin n\theta + \bar{A}_{3k}{}^v \sin(n+1)\theta] \sin(m_k\pi x/L) \quad (17)$$

$$\Psi_x = \sum_k [\bar{A}_{4k}{}^x \cos(n-1)\theta + \bar{A}_{5k}{}^x \cos n\theta + \bar{A}_{6k}{}^x \cos(n+1)\theta] \cos(m_k\pi x/L)$$

$$\Psi_\theta = \sum_k [\bar{A}_{1k}{}^\theta \sin(n-1)\theta + \bar{A}_{2k}{}^\theta \sin n\theta + \bar{A}_{3k}{}^\theta \sin(n+1)\theta] \sin(m_k\pi x/L)$$

This set of modal functions satisfies the geometric boundary conditions w, $w_{,x}$, v, and $\Psi_\theta = 0$ at $x = 0$ and L. However, the force conditions are not fully satisfied.

The additional cos θ term which appears in the bending portion of Eq. (A1) results in a θ coupling of the nth term with the $(n - 1)$ and $(n + 1)$ terms. Still further coupling is present; for example, the $(n + 1)$ term is coupled with the $(n + 2)$ term in addition to the nth term. This secondary coupling has been neglected in the set of functions of Eqs.

Table 1 Geometric and material characteristics of the Ref. 5 sandwich cylinders with glass-fiber/epoxy facings and aluminum honeycomb cores

Dimensions			
R = 21.94 in.	L = 72.0 in.	h = 0.15 in.	t = 0.01 in.
Facing material properties			
$E_x = 2.28 \times 10^6$ psi	$E_\theta = 3.14 \times 10^6$ psi	$G_{x\theta} = 0.416 \times 10^6$ psi	$\nu_{x\theta} = 0.13$
Core properties			
G_{xz} = 32,000 psi		$G_{\theta z}$ = 18,300 psi	

Table 2 Test results of Ref. 19 for sandwich cylinder with aluminum facings and 0.200-in.-depth aluminum honeycomb cores

Facing thickness, in.	Failure stress, ksi	Theoretical buckling stress, ksi
0.0143	56.7	69.8
0.0149	48.7	69.8
0.0150	48.3	69.8

(17), to keep the size of the resulting matrices manageable. Substituting Eqs. (17) into the total energy expression, Eq. (A1), results in an equation of the form of Eq. (14).

A computer program was written for solving Eq. (15). This program includes subroutines for performing the various θ and x integrations required. Numerical methods were used for several of the integrations for which closed-form solutions did not exist. In the matrices of Eq. (15), the rows represent $(\partial/\partial \bar{A}_{ij}{}^k)$ and the columns represent the coefficient of $\bar{A}_{lm}{}^n$

The energy expression, Eq. (A1), was checked utilizing a set of simple-support modal functions and axial compression loading and the results were compared with those predicted by the Donnell-type theory of Bert et al.[5] The Donnell-type theory predicted a critical load of 25,285 psi for the Table 1 cylinder; as expected, the Love's first-approximation theory, Eq. (A1), predicted a slightly lower value (24,869 psi). These simple-support analyses were found also to be helpful in determining the approximate range of values for the longitudinal modal functions.

8. Evaluation of the Theory

For clamped boundary conditions, the theory was evaluated for axial compression, pure bending, and combined axial compression and bending. Although clamped boundary conditions are probably the most feasible from a test standpoint, there are very few experimental data available. For clamped boundary conditions, the present theory predicts a critical stress of 25,350 psi for axial compression loading of the Ref. 5 cylinders, Table 1, as expected. This is slightly higher than that predicted for simple supports. The analytical results were compared with the experimental results reported in Ref. 5 for orthotropic sandwich cylinders loaded in pure bending and the experimental results of Ref. 19 for axially compressed sandwich cylinders with isotropic facings. The effects of plasticity were accounted for in comparing with the data of Ref. 19 by substituting $(E_sE_t)^{1/2}$ for the longitudinal elastic moduli. This is the same procedure suggested in Ref. 19 and was obtained from Ref. 20. The variation in Poisson's ratio was accounted for by using the following relationship proposed by Nadai[21]:

$$\nu = (1/2) - [(1/2) - \nu_e]\,(E_s/E) \tag{18}$$

It is realized that more rigorous methods exist for taking plasticity into account; however, the complicated form of the solution makes it advantageous to use a simple procedure such as outlined above. This procedure can result in several interactions since it is necessary to obtain the secant and tangent moduli at the critical stress.

8.1 Axial Compression Loading

In Ref. 19 axial compression tests were conducted on sandwich cylinders with 7178-T6 bare aluminum-alloy facings and Hexcel (1/8)-5052-0.0015P aluminum-alloy honeycomb core. The cylinders had an over-all length of 60.0 in. and an inside diam of 55.0 in. The test results are shown in Table 2.

The theoretical values are also shown in Table 2. No other analyses exist for clamped sandwich cylinders. The simple-support analysis of Ref. 5 and 8 provides a slightly lower estimate, as would be expected.

8.2 Pure Bending Loading

In Ref. 5 are described pure bending tests which were conducted on an orthotropic sandwich cylinder. The material and geometric properties are given in Table 1.

The experimental critical buckling stress was 14,100 psi. The computer program gave a value for the critical

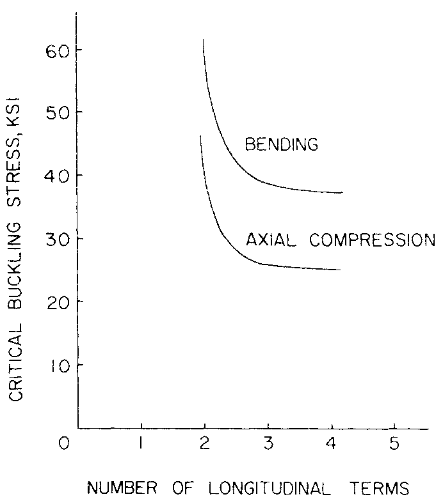

Fig. 3 Convergence for cylinders of Ref. 5.

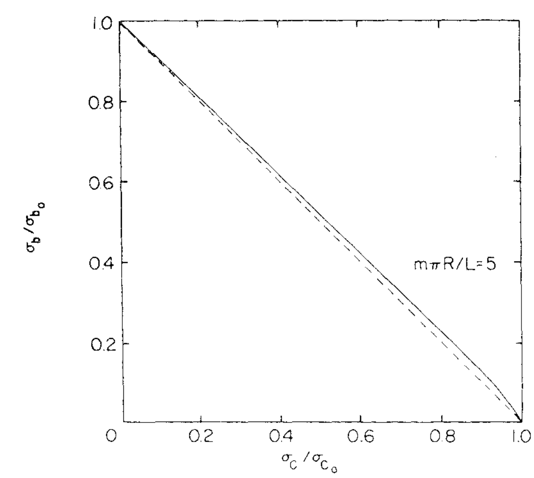

Fig. 4 Axial compression and bending interaction curve calculated for cylinders of Ref. 5.

stress of 37,550 psi, using four terms in the assumed longitudinal modal function. Figure 3 shows the convergence with increasing number of assumed longitudinal terms for the Ref. 5 cylindrical specimens for both axial compression and bending.

In the past, buckling under pure bending loading has been predicted by the assumption that failure occurs when the maximum bending stress reaches the critical value for axial compression buckling, i.e., $\sigma_{bo} \approx \sigma_{co}$. As seen in Fig. 3, according to the present analysis, this would be a very conservative estimate.

Apparently the only existing analysis available for buckling of isotropic sandwich cylinders under pure bending is due to Gellatly and Gallagher.[22] They obtained values of σ_{bo}/σ_{co} ranging from 1.013 to 1.043 for simply supported cylinders. The results of the present analysis also can be compared with the isotropic thin-wall-shell analysis of Ugural and Cheng,[24] both of which assumed simply supported ends. The comparison was made using an equivalent thin-wall-shell wall thickness for the sandwich construction of $2(3^{1/2})\,(h + t)$. This gives an equivalent thickness of 0.5545 in. and a R/thickness ratio of 40 for the cylinder of Ref. 5. An $m\pi R/L$ ratio of 5 (based on the strongest longitudinal term in the eigenvector) was found for these same cylinders. For the isotropic case, it was reported in Ref. 23 that the ratio of critical stress in bending to critical stress in axial compression, σ_{bo}/σ_{co}, for a wavelength parameter of 5 and a R/thickness ratio of 100, was approximately 1.20. In Ref. 24 a value of approximately 1.35 for this ratio was mentioned. For the Ref. 5 cylinders, the present analysis gives a value of 1.48 for this ratio.

8.3 Combined Axial Compression and Bending Loading

There are no experimental data available for this combined loading case; however, the interaction curve calculated for the Ref. 5 cylinders shown in Fig. 4 is of the same general appearance as the isotropic cylinder interaction curves reported in Ref. 23. Figure 5 shows the interaction curves from Ref. 23 for a R/thickness ratio of 100 and for three different values of the wavelength parameter, $(m\pi R)/L$. For a radius/thickness ratio of 100, the σ_{bo}/σ_{co} ratio reaches a minimum (minimum buckling curve) for a wavelength parameter of 17. It is noted that both interaction curves, Figs. 4 and 5, are of the same general form. The set of interaction curves given in Fig. 5 is typical of those presented in Ref. 23 for different radius/thickness ratios.

9. Closure

The general theory presented here for general instability buckling under axial compression, bending, and any combination of them can be applied to cases involving different assumed modal shapes.

For bending, the results are considerably higher than those of Ref. 22 for the isotropic sandwich cylinders; however, the results are similar to those predicted in Refs. 23 and 24 for the isotropic case.

Appendix—Total Energy Expression

$$\text{Total Energy} = \int_x \int_\theta -\lambda [\,\bar{V}_1 w_{,x}{}^2 + \bar{V}_2 \zeta w_{,x}{}^2 \cos\theta] +$$
$$\{A_{13}R(w_{,x}{}^2 + 2\,w_{,x}\Psi_x + \Psi_x{}^2) + A_{14}(R\Psi_\theta{}^2 + \zeta v^2 + \zeta w_{,\theta}{}^2 -$$
$$2\zeta v w_{,\theta} - 2\,\Psi_\theta v + 2\Psi_\theta w_{,\theta}) + R[A_1(u_{,x}{}^2 + h^2\Psi^2{}_{x,x})$$
$$+ A_4 w_{,xx}{}^2 - A_7\Psi_{x,x}w_{,xx}] + \zeta[A_2(v_{,\theta}{}^2 + h^2\Psi^2{}_{\theta,\theta} + w^2 +$$
$$2\,v_{,\theta}w) + A_5(\zeta^2 v_{,\theta}{}^2 + h^2\zeta^2\Psi^2{}_{\theta,\theta} + \zeta^2 w_{,\theta\theta}{}^2 -$$

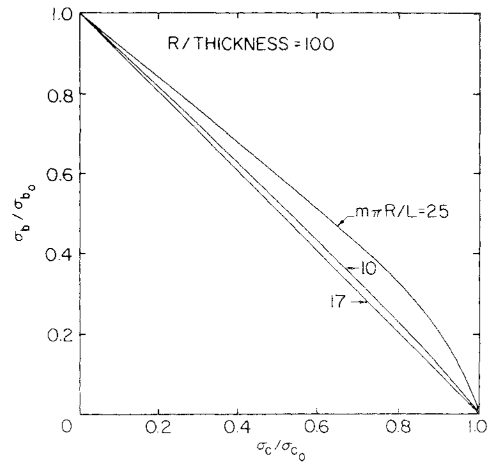

Fig. 5 Typical axial compression and bending interaction curves for isotropic cylinders of Ref. 23.

$$2\,\zeta^2\,v_{,\theta}w_{,\theta\theta}) + A_9(2\,\zeta v_{,\theta}\Psi_{\theta,\theta} - \zeta\Psi_{\theta,\theta}w_{,\theta\theta}$$
$$+ \zeta\Psi_{\theta,\theta}w)] + A_3(u_{,x}\;v_{,\theta} + u_{,x}w) + A_6(\zeta w_{,xx}w_{,\theta\theta} -$$
$$\zeta w_{,xx}v_{,\theta}) + A_8\,(\zeta u_{,x}\Psi_{\theta,\theta} - w_{,xx}\Psi_{\theta,\theta} - \zeta\Psi_{x,x}w_{,\theta\theta} +$$
$$\zeta v_{,\theta}\Psi_{x,x}) + A_{15}\Psi_{x,x}\Psi_{\theta,\theta} + \zeta[A_{10}(u_{,\theta}{}^2 +$$
$$h^2\Psi^2{}_{x,\theta} + R^2 v_{,x}{}^2 + h^2R^2\Psi_{\theta,x}{}^2 + 2Ru_{,\theta}v_{,x} +$$
$$2h^2R\Psi_{x,\theta}\Psi_{\theta,x}) + A_{11}(-2\Psi_{x,\theta}w_{,x\theta} - 2R\Psi_{\theta,x}w_{,x\theta} + u_{,\theta}\Psi_{\theta,x} +$$
$$v_{,x}\Psi_{x,\theta} + 2Rv_{,x}\Psi_{\theta,x}) + A_{12}(4w_{,x\theta}{}^2 + v_{,x}{}^2$$
$$+ h^2\Psi_{\theta,x}^2 - 4v_{,x}w_{,x\theta}]\}d\theta dx \quad \text{(A1)}$$

where

$A_1 = 2t\bar{E}_x$ $\qquad A_2 = 2t\bar{E}_\theta$

$A_3 = 4t\bar{E}_x\nu_{\theta x}$ $\qquad A_4 = (8/3)t^3\bar{E}_x$

$A_5 = (8/3)t^3\bar{E}_\theta$ $\qquad A_6 = (16/3)t^3\bar{E}_x\nu_{\theta x}$

$A_7 = 4ht^2\bar{E}_x$ $\qquad A_8 = 4ht^2\bar{E}_x\nu_{\theta x}$

$A_9 = 4ht^2\bar{E}_\theta$ $\qquad A_{10} = 2tG_{x\theta}$

$A_{11} = 4ht^2G_{x\theta}$ $\qquad A_{12} = (8/3)t^3G_{x\theta}$

$A_{13} = hK_xG_{zx}$ $\qquad A_{14} = hK_\theta G_{\theta z}$

$A_{15} = 4h^2t\bar{E}_x\nu_{\theta x}$

$\bar{V} = P'/(4\pi)$ $\qquad \bar{V}_2 = -M'/(2\pi)$

(A2)

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Second-Order Optimality Conditions for the Bolza Problem with Both Endpoints Variable

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Three sets of sufficient conditions for a weak minimum are derived for a nonsingular form of the Bolza problem of variational calculus, expressed in control notation. Both endpoints are assumed to be variable, subject to a set of equality constraints. The end conditions are assumed to be separated. Both specified and unspecified end times are considered. Conditions involving the backward and/or forward integration of a matrix Riccati equation are obtained. Several of these conditions are related to classical conditions involving conjugate points or focal points, but are more easily implemented in realistic optimization problems. Other conditions appear to have no direct classical counterparts. Necessary conditions for a weak minimum differ only slightly from the sufficient conditions. These conditions are easier to apply than another recently proposed set of second-order necessary conditions. The results of this paper are applied to two simple problems.

Introduction

THE form of the Bolza problem considered in this paper may be expressed in modern control notation in the following manner.

Among the set of all continuous unbounded n-dimensional state variable functions $x(t)$ and m-dimensional control variable functions $u(t)$ satisfying nonlinear differential equations of the form

$$\dot{x}(t) = f(x,u,t), \quad t_o \leq t \leq t_f \tag{1}$$

as well as $r_o \leq n$ initial constraints of the form

$$Y_o(x_o,t_o) - y_o = 0 \tag{2}$$

and $r_f \leq n$ terminal constraints of the form

$$Y_f(x_f,t_f) - y_f = 0 \tag{3}$$

find the set that will minimize the scalar performance index

$$J = g_o(x_o,t_o) + g_f(x_f,t_f) + \int_{t_o}^{t_f} L(x,u,t)dt \tag{4}$$

Problems of this form are said to have separated end conditions.[1] The initial and terminal times are assumed to be unspecified. y_o and y_f are vectors of constants. The functions f, L, Y, and g are assumed to be twice continuously differentiable with respect to their arguments. A dot above a variable denotes differentiation with respect to time. The subscripts o and f denote evaluation of a quantity at t_o and t_f, respectively. Many optimization prob-

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